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One-dimensional model for the Rijke tube

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We develop a simple model in which longitudinal, compressible, unsteady heat transfer between heater and gas is computed in the small-Mach-number limit. This calculation is used to determine the transfer function of the heater, which plays an important role in the stability limits of the thermoacoustic instability of the Rijke tube. The transfer function is determined analytically in the limit of small expansion parameter γ, and numerically for γ of order unity. In the case $\rho\mu/c_p$ = constant, an analytical solution can be found.

1. Introduction

We consider the Rijke phenomenon, i.e. the spontaneous thermal-acoustic oscillation that can appear when a heated gauze is placed in a tube in which a gas flows. A similar phenomenon occurs when a flame is anchored on a grid in the tube. All these phenomena are grouped together under the name 'gauze tones' (for a review, see for instance Strehlow 1979, Merk 1956 and Putman 1964). A general criterion for the acoustic instability was developed by Lord Rayleigh. When heat is released locally and periodically in a gaseous medium, an acoustic oscillation is amplified if the oscillating components of pressure and released heat flux are in phase. The problem remains to determine the relative phase between the pressure field and the released heat flux, which is in general a difficult problem of heat transfer.

Since the heating process is localized, the heat transfer in the gas occurs in a small region around the heater. Outside this region, the temperature is uniform and flow disturbances are acoustic waves. When such a wave comes across the heater region, the fluctuating incoming flow causes a heat flux variation and, thus, a variation of gas density inside this region. As a result, because of mass conservation, a jump in the acoustic velocity field across this region occurs, which appears as a boundary condition for the acoustic field in the tube, applied at the heater position. The other boundary condition is that pressure disturbances are equal on both sides of the heater, since the flow is at small Mach number. Thus all the characteristics of the heat transfer between heater and gas are contained in the jump in the acoustic velocity field, whose relative value is called the transfer function.

The calculation of the transfer functions is laborious since it involves the calculation of unsteady heat transfer in a compressible flow of gas. In previous works, in order to make the problem tractable, many simplifying assumptions have been adopted, in particular that the flow in the heater region is incompressible. Thus the feedback of the heat flux variation on the mass flow rate disturbance is neglected. Lighthill (1954) approximates the heater by a straight cylinder and performs a boundary-layer calculation. The transfer function is computed at low and high frequency. Cole & Roshko (1956), and more recently Bayly (1985), consider the same configuration, but in the low-Reynolds-number limit, and use the Oseen approximation to compute the transfer function. In this approximation, transport of heat by conduction is the dominant process near the body, and transport by heat convection only becomes important near infinity, where the fluctuating incoming flow is given. A similar calculation was done by Carrier (1955), in which the heater is a ribbon without thickness but with width d in the direction of the flow.

Thus, it seems useful to develop a sufficiently simple model in which compressible effects are taken into account and in which the transfer function can be computed using controlled approximations. First, we present the model studied and describe the approximations that are made. Then, we determine the transfer function. It is computed analytically in the limit of small expansion parameter $\gamma = (T_2 - T_1)/T_1$, where T_1 and T_2 are the temperatures of fresh and hot gases respectively, and numerically for values of γ of order unity. In the case $\rho\mu/c_p$ = constant, where ρ, μ, c_p are the density of the gas, the heat conductivity and the specific heat per unit mass, respectively, an analytical solution can be found for any value of γ.

2. The model

The simplest heating apparatus one can choose is a grid perpendicular to the tube axis, infinitely thin and conductive, heated at constant temperature T_2. It is assumed to be of infinite dimension in the transverse direction, so that only longitudinal heat conduction need be considered. Thus, it appears as a discontinuity surface for the monodimensional fields of temperature, gas velocity and pressure, at which boundary conditions are applied. The equations of conservation of mass, momentum and energy are simply

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0, \tag{1}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = -\frac{\partial p}{\partial x} + (\eta + \eta')\frac{\partial^2 u}{\partial x^2}, \tag{2}$$

$$\rho c_p\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x}\right) = \frac{\partial}{\partial x}\left(\mu\frac{\partial T}{\partial x}\right) + \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x}. \tag{3}$$

Here, u is the longitudinal gas velocity, p the pressure, T the temperature, and η and η' the first and second coefficients of dynamical viscosity. Furthermore p satisfies the ideal gas law, i.e.

$$p = \frac{\rho R T}{M_0}, \tag{4}$$

where M_0 is the molar mass.

In order to discuss the boundary conditions that one must apply at the heater location ($x = 0$), assume that the grid is built with thin wires of diameter of order b, the mesh size being of the same order. The flow is perturbed around the grid on the same distance and, because of viscous friction, a pressure jump will appear across the grid, of order

$$p_- - p_+ = \delta p_{\text{grid}} \approx \frac{\eta u_{\text{grid}}}{b}, \tag{5}$$

where u_{grid} is the mean flow velocity at $x = 0$. The subscripts $-$ and $+$ mean the pressure on the upstream and downstream sides of the heater, respectively. The grid can be considered as a discontinuity surface for the outer field if b is much smaller than the diffusive length $l \approx \mu/\rho c_p U$, where U is the velocity of the flow far upstream from the grid. As a result, the diffusive time $b^2\rho c_p/\mu$ on a distance of order b is small compared with the transit time across the grid, b/U. Thus, the gas has time to reach the grid temperature, T_2, so that, on the diffusive scale, temperatures on both sides of the grid can be considered as equal to T_2, i.e.

$$T_- = T_+ = T_2. \tag{6}$$

Furthermore, conservation of mass flux across the heater leads to

$$\rho_- u_- = \rho_+ u_+. \tag{7}$$

The configuration considered is that of the classical Rijke tube experiment: the gas, initially at temperature T_1 (density ρ_1) flows at constant rate $m = \rho_1 U$ across the grid.

3. The stationary solution

As can be deduced from (1)–(4), the stationary profiles of T, ρ, u and p satisfy

$$m = \bar{\rho}\bar{u}, \tag{8}$$

$$m\frac{\mathrm{d}\bar{T}}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\nu(\bar{T})\frac{\mathrm{d}\bar{T}}{\mathrm{d}x}\right) + \frac{\bar{u}}{c_p}\frac{\mathrm{d}\bar{p}}{\mathrm{d}x}, \tag{9}$$

$$m\frac{\mathrm{d}\bar{u}}{\mathrm{d}x} = -\frac{\mathrm{d}\bar{p}}{\mathrm{d}x} + (\eta+\eta')\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}x^2}, \tag{10}$$

and

$$\bar{\rho}\bar{T} = \frac{M_0}{R}\bar{p}, \tag{11}$$

where, for simplicity $\nu \equiv \mu/c_p$. For a perfect gas, ν behaves like $T^{\frac{1}{2}}$.

The flow is assumed to have small Mach number, so that one can expand the steady solution as

$$\frac{\bar{T}}{T_1} = \theta_0 + M^2\theta_1 + \dots, \tag{12}$$

$$\frac{\bar{\rho}}{\rho_1} = r_0 + M^2 r_1 + \dots, \tag{13}$$

$$\frac{\bar{p}}{P_1} = \pi_0 + M^2\pi_1 + \dots, \tag{14}$$

and

$$\frac{\bar{u}}{U} = u_0 + M^2 u_1 + \dots, \tag{15}$$

where $P_1 = \rho_1 R T_1/M_0$ is the pressure of the fresh gas. At zero order in the small-Mach-number limit, one obtains, using (8)–(11), the following equations:

$$r_0 u_0 = 1, \tag{16}$$

$$\frac{\mathrm{d}\theta_0}{\mathrm{d}\xi} = \frac{\mathrm{d}^2\theta_0}{\mathrm{d}\xi^2}, \tag{17}$$

$$\frac{\mathrm{d}\pi_0}{\mathrm{d}\xi} = 0, \tag{18}$$

and $$r_0\theta_0 = 1, \tag{19}$$

where the new spatial coordinate ξ is defined by

$$\mathrm{d}\xi = \frac{m}{\nu(T_1\theta_0)}\mathrm{d}x, \tag{20}$$

with $\xi(0) = 0$.

Equation (17) can be integrated by using boundary condition (6) to give the leading order of the steady solution

$$\left.\begin{aligned}\bar{T}(\xi) &= T_1+(T_2-T_1)\,\mathrm{e}^{\xi}+O(M^2) \quad && \xi < 0,\\ \bar{T}(\xi) &= T_2+O(M^2) && \xi > 0,\end{aligned}\right\} \tag{21}$$

from which $\bar{\rho}$ and $\bar{u}$ are easily derived:

$$\bar{\rho} = \frac{\rho_1 T_1}{\bar{T}}+O(M^2), \tag{22}$$

$$\bar{u} = \frac{m}{\bar{\rho}}+O(M^2). \tag{23}$$

In particular, as can be deduced from (18), the pressure, in this approximation, is uniform on both sides of the grid. It follows that the jump of pressure between the fresh and hot regions is $p_1-p_2 = \delta p_{\mathrm{grid}}$. It can be seen from (5) that the relative jump of pressure is small compared with 1, but large compared with M^2 since b is small compared with the diffusive length.

4. The transfer function

4.1. *Scalings and different regions*

When a sound wave of frequency ω and wavelength λ encounters the grid, stationary profiles of the different physical quantities are perturbed. Velocity and pressure variations will vary on two different lengthscales in two different regions. The ratio l/λ can be written as $l/\lambda = M\omega\tau$, were $\tau = l/U$ is the diffusive time. Thus, if $\omega\tau$ is assumed of order unity, which is the case in experiments, l becomes very small compared with λ when the Mach number is small.

4.1.1. *The acoustic region*

Far away from the grid, in both directions, the steady temperature is uniform and disturbances are regular acoustic waves, i.e. they vary on the acoustic wavelength $\lambda = c/\omega$. Such waves are characterized by a modulation of pressure p' and gas velocity u', and the problem is to determine how such quantities are modified across the heater region. In (1)–(4), lengths are scaled by λ. At zero order in the expansion at small Mach number, these equations reduce to the usual equations for acoustic disturbances propagating in a fluid at rest, i.e.

$$\frac{\partial \rho'}{\partial t}+\bar{\rho}\frac{\partial}{\partial x}(u') = 0, \tag{24}$$

$$\bar{\rho}\frac{\partial u'}{\partial t} = -\frac{\partial p'}{\partial x}, \tag{25}$$

$$\bar{\rho}c_p\frac{\partial T'}{\partial t} = \frac{\partial p'}{\partial t}, \tag{26}$$

and
$$p' = \frac{R}{M_0}(\rho'\bar{T}+\bar{\rho}T'). \tag{27}$$

From these relations follows the usual relation between pressure and velocity in a progressive acoustic wave, i.e. $p' = \rho c u'$.

4.1.2. *The diffusion region*

Inside the diffusion region, i.e. where the steady solution is not uniform, quantities vary on a scale l, the diffusive length, which is much smaller than λ. One begins to evaluate the jump of pressure disturbance across the diffusion region by integrating (2) over a distance of order l. The contributions of the inertia terms $\rho(\partial u/\partial t)$ and $\rho u\, \partial u/\partial x$ to the jump of pressure are respectively

$$|\delta p_1'| \sim \rho\omega u' l \tag{28}$$

and
$$|\delta p_2'| \sim m u'. \tag{29}$$

Thus, the corresponding relative jumps are

$$\frac{|\delta p_1'|}{p'} \sim \frac{l}{\lambda}, \tag{30}$$

i.e. of order M, since the product $\omega\tau$ was assumed of order unity, and

$$\frac{|\delta p_2'|}{p'} \sim M. \tag{31}$$

The contribution of viscous stresses in the gas can be evaluated as

$$|\delta p_3'| \sim \frac{\eta u'}{l}, \tag{32}$$

and thus
$$\frac{|\delta p_3'|}{p'} \sim \frac{\eta}{\rho c l} \sim M, \tag{33}$$

since the Prandtl number is of order unity.

An additional contribution from viscous stresses arises in the friction of the gas through the mesh of the grid. From relation (5), one has simply

$$|\delta p_4'| \sim \frac{\eta}{b} u', \tag{34}$$

and thus
$$\frac{|\delta p_4'|}{p'} \approx \frac{\eta}{\rho b c} \approx M\frac{l}{b} \ll 1, \tag{35}$$

since b is assumed very large compared with the mean free path of the molecules in the gas. Of all these contributions, the last one dominates, but is still negligible compared with 1. Thus, when the Mach number of the flow is small, the total relative jump of pressure disturbance across the diffusion region can be neglected. Thus, in a first approximation, a first boundary condition for the acoustic field that must be applied is

$$p_1' = p_2' + O\left(M\frac{l}{b}\right). \tag{36}$$

As a result, the diffusion region can be considered as isobaric, i.e. the product ρT is constant. Similarly, in (3), the contribution of the terms $\partial p/\partial t$ and $u\,\partial p/\partial x$ are respectively of order $\omega p' \approx \omega\rho c u'$ and mUu'/l. The order of magnitude of the term $\rho u c_p \,\partial T/\partial x$ is $\rho u'\gamma c^2/l$. Thus, the relative magnitudes of the pressure terms in (3) are respectively $M\omega\tau/\gamma$ and M^2/γ, and can be neglected if $\gamma \gg M$, since $\omega\tau$ was assumed of order unity. In these conditions the system of equations (1)–(4) simplifies to

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}=0, \tag{37}$$

$$\rho c_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}\right)=\frac{\partial}{\partial x}\left(\mu\frac{\partial T}{\partial x}\right), \tag{38}$$

and

$$\rho T=\rho_1 T_1, \tag{39}$$

with the boundary conditions (6) and (7).

Let $\delta\rho u(\xi)\exp(\mathrm{i}\omega t)$, $\delta\rho(\xi)\exp(\mathrm{i}\omega t)$ and $\theta(\xi)\exp(\mathrm{i}\omega t)$ be the disturbances of mass flow rate, density and temperature of the steady solutions given by (21)–(23). From (37)–(39) and (20), one deduces

$$\mathrm{i}\omega\,\delta\rho+\frac{m}{\nu(\bar T)}\frac{\mathrm{d}}{\mathrm{d}\xi}(\delta\rho u)=0, \tag{40}$$

$$\bar\rho\mathrm{i}\omega\theta+\frac{m^2}{\nu(\bar T)}\frac{\mathrm{d}\theta}{\mathrm{d}\xi}-\frac{m^2}{\nu(\bar T)}\frac{\mathrm{d}^2\theta}{\mathrm{d}\xi^2}-\frac{m^2}{\nu(\bar T)}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\theta\frac{1}{\nu}\frac{\mathrm{d}\nu}{\mathrm{d}T}\bigg|_{T=\bar T}\frac{\mathrm{d}\bar T}{\mathrm{d}\xi}\right)=-\delta\rho u\frac{m}{\nu(\bar T)}\frac{\mathrm{d}\bar T}{\mathrm{d}\xi}, \tag{41}$$

$$\delta\rho\bar T+\bar\rho\theta=0. \tag{42}$$

By using (41) and (42), one obtains

$$\theta=\frac{1}{\mathrm{i}\omega}\frac{\bar T}{\bar\rho}\frac{m}{\nu(\bar T)}\frac{\mathrm{d}}{\mathrm{d}\xi}(\delta\rho u), \tag{43}$$

and from (41),

$$\frac{\mathrm{d}^3\delta\rho u}{\mathrm{d}\xi^3}-\frac{\mathrm{d}^2\delta\rho u}{\mathrm{d}\xi^2}\left(1-2\frac{\bar\rho\bar\nu\mathrm{d}}{\bar T\mathrm{d}\xi}\left(\frac{\bar T}{\bar\rho\bar\nu}\right)-\frac{1}{\nu}\frac{\mathrm{d}\nu}{\mathrm{d}T}\bigg|_{T=\bar T}\frac{\mathrm{d}\bar T}{\mathrm{d}\xi}\right)$$
$$-\frac{\mathrm{d}\delta\rho u}{\mathrm{d}\xi}\left(\mathrm{i}\omega\frac{\bar\rho\nu}{m^2}+\frac{\bar\rho\nu}{\bar T}\left(\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{\bar T}{\bar\rho\nu}\right)-\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(\frac{\bar T}{\bar\rho\nu}\right)\right)\right.$$
$$\left.-\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{1}{\nu}\frac{\mathrm{d}\nu}{\mathrm{d}T}\bigg|_{T=\bar T}\frac{\mathrm{d}\bar T}{\mathrm{d}\xi}\right)-\frac{1}{\nu}\frac{\mathrm{d}\nu}{\mathrm{d}T}\bigg|_{T=\bar T}\frac{\mathrm{d}\bar T}{\mathrm{d}\xi}\frac{\bar\rho\nu}{\bar T}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{\bar T}{\bar\rho\nu}\right)\right)-\mathrm{i}\omega\frac{\bar\rho\nu}{m^2\bar T}\frac{\mathrm{d}\bar T}{\mathrm{d}\xi}\delta\rho u=0. \tag{44}$$

Far from the heater in the fresh gases, $\delta\rho u=\delta\rho u_{-\infty}$, the mass flow rate disturbance of the incoming sound wave. At $\xi=0$, $\theta=0$ since the grid is at constant temperature T_2, and thus, $\mathrm{d}(\delta\rho u)/\mathrm{d}\xi=0$ as can be deduced from (43).

4.2. *The transfer function*

Behind the grid, in the hot gases, disturbances are sound waves and $\delta\rho u(0)$ is the mass flow rate disturbance of the outgoing sound wave. The transfer function is defined as

$$Tr=\frac{\delta u(0)-\delta u_{-\infty}}{\delta u_{-\infty}}. \tag{45}$$

Since $\delta\rho u=\bar\rho\delta u+\delta\rho\bar u$, and since for a sound wave, $\delta\rho u\sim c\delta\rho$, one can neglect $\delta\rho\bar u$

compared with $\delta\rho u$ at both $\xi = -\infty$ and $\xi = 0$. Thus the transfer function can be rewritten as

$$Tr = \frac{\rho_1}{\rho_2}\frac{\delta\rho u(0)}{\delta\rho u_{-\infty}} - 1. \tag{46}$$

In order to progress further, one takes $\nu(T) \sim T^{\frac{3}{2}}$, as is the case for a perfect gas. Introducing the dimensionless variable $\phi = \delta\rho u/\delta\rho u_{-\infty}$ and the expansion parameter $\gamma = (T_2 - T_1)/T_1$, (44) becomes

$$\frac{d^3\phi}{d\xi^3} - \frac{d^2\phi}{d\xi^2}\left(1 - \frac{7}{2}\frac{\gamma e^{\xi}}{1+\gamma e^{\xi}}\right) - \frac{d\phi}{d\xi}\left(\frac{i\omega\tau}{(1+\gamma e^{\xi})^{\frac{1}{2}}} - \frac{3}{2}\left(\frac{\gamma e^{\xi}}{1+\gamma e^{\xi}}\right)^2 - \frac{1}{2}\frac{\gamma e^{\xi}}{(1+\gamma e^{\xi})^2}\right) - i\omega\tau\frac{\gamma e^{\xi}}{(1+\gamma e^{\xi})^{\frac{3}{2}}}\phi = 0, \tag{47}$$

with boundary conditions

$$\phi(-\infty) = 1, \quad \phi'(0) = 0. \tag{48}$$

Here $\tau = \rho_1 \nu_1/m^2$ is the diffusive time in the fresh gases. When a solution $\phi(\xi)$ of (47) satisfying the boundary conditions (48) is determined, the transfer function is computed by using (46), i.e.

$$Tr = (1+\gamma)\,\phi(0) - 1. \tag{49}$$

4.2.1. *Asymptotic expansion for small values of the parameter* γ

Equation (47) cannot be solved analytically for any value of γ. But an asymptotic expansion of the solution can be performed for small values of γ, but much larger than M, and values of $\omega\tau$ of order 1.

One expands $\phi(\xi)$ as

$$\phi(\xi) = \phi_0(\xi) + \gamma\phi_1(\xi) + o(\gamma), \tag{50}$$

and the transfer function Tr as

$$Tr = Tr_0 + \gamma\, Tr_1 + o(\gamma). \tag{51}$$

At zero order in γ, $\phi_0(\xi)$ satisfies

$$\frac{d^3\phi_0}{d\xi^3} - \frac{d^2\phi_0}{d\xi^2} - i\omega\tau\frac{d\phi_0}{d\xi} = 0, \tag{52}$$

from which one deduces

$$\frac{d\phi_0}{d\xi}(\xi) = A_0 \exp(\chi_+\xi) + B_0 \exp(\chi_-\xi), \tag{53}$$

where

$$\chi_\pm = \frac{1 \pm (1+4i\omega\tau)^{\frac{1}{2}}}{2}. \tag{54}$$

Here, $\omega\tau$ is real and only χ_+ has a positive real part. Thus, B_0 and A_0 are equal to zero since $d\phi_0(0)/d\xi = 0$. Because $\phi_0(-\infty) = 1$ one can deduce

$$\phi_0(\xi) = 1 \tag{55}$$

and thus, from relation (49), $Tr_0 = 0$.

At first order in γ, $\phi_1(\xi)$ satisfies

$$\frac{d^3\phi_1}{d\xi^3} - \frac{d^2\phi_1}{d\xi^2} - i\omega\tau\frac{d\phi_1}{d\xi} = i\omega\tau\, e^{\xi}, \tag{56}$$

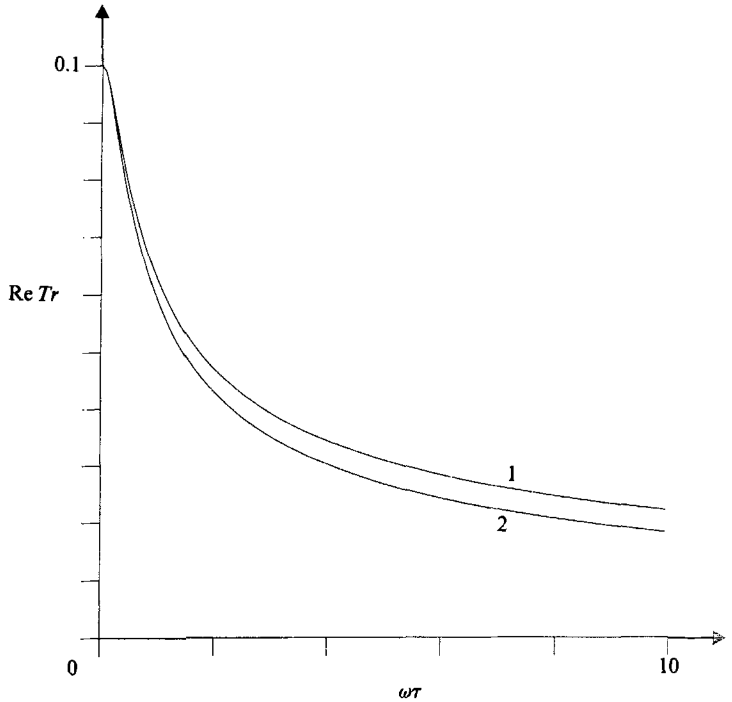

FIGURE 1. Real part of the transfer function for $\gamma = 0.1$: curve 1, asymptotic transfer function (60); curve 2, numerical result.

and thus
$$\frac{d\phi_1}{d\xi}(\xi) = A_1 \exp(\chi_+ \xi) - e^{\xi}, \tag{57}$$
from which, one deduces, by using the boundary conditions,
$$\phi_1(\xi) = \frac{1}{\chi_+} \exp(\chi_+ \xi) - e^{\xi}. \tag{58}$$
The corresponding contribution to the transfer function is, by using (49),
$$Tr_1 = \chi_+^{-1}. \tag{59}$$
Thus, up to the first order in γ, the transfer function is
$$Tr(\omega\tau) = \frac{2\gamma}{1+(1+4i\omega\tau)^{\frac{1}{2}}}, \tag{60}$$
whose real part and imaginary part are drawn on figures 1 and 2 (curves 1).

The right-hand side of (56) comes from the last term of the left-hand side of (47). At first order in γ, variations of heat conductivity appear unimportant since the dominant term comes from the expression for $dT/d\xi$.

4.2.2. *Thermoacoustic instability*

First, one recalls the equation for the acoustic eigenmodes of the tube. The tube is of length L, with free extremities located at $x = -rL$ and $x = (1-r)L$. The heater is at $x = 0$. One neglects all damping effects, i.e. losses by acoustic radiation at the free extremities and losses by transverse heat conduction and friction at the tube

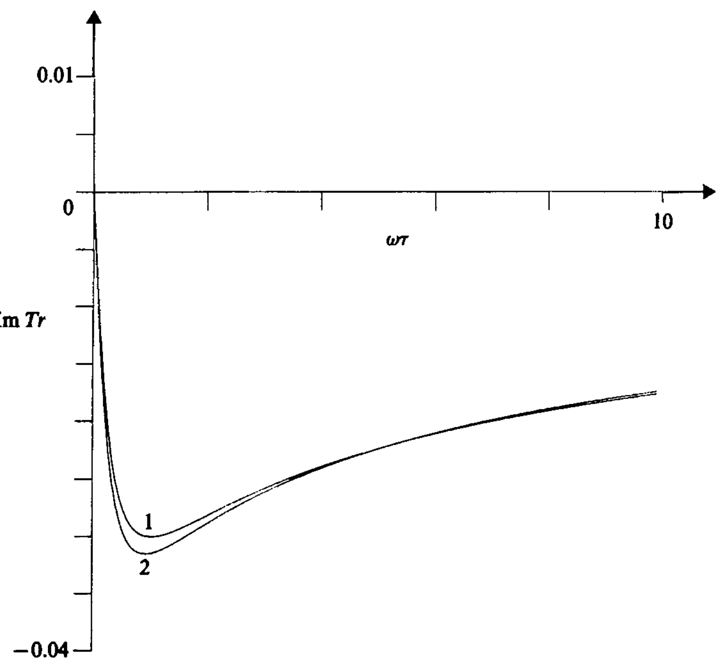

FIGURE 2. Imaginary part of the transfer function for $\gamma = 0.1$: curve 1, asymptotic transfer function (60); curve 2, numerical result.

walls. In these conditions, as can be deduced from (24)–(27), the acoustic field at the heater satisfies

$$u_1(0) = \frac{p(0)}{\rho_1 c_1}\cot(rX), \quad u_2(0) = \frac{p(0)}{\rho_2 c_2}\cot\left((1-r)\frac{c_1}{c_2}X\right), \tag{61}$$

where $X = \omega L/c_1$ is the dimensionless frequency of the acoustic wave.

By using boundary conditions (36) and (45) at the heater, one finally obtains the eigenmode equation as

$$\sin(rX)\cos\left((1-r)\frac{c_1}{c_2}X\right)+\frac{\rho_2 c_2}{\rho_1 c_1}(1+Tr)\sin\left((1-r)\frac{c_1}{c_2}X\right)\cos(rX) = 0. \tag{62}$$

When the transfer function Tr takes small values, as it is the case for instance when the expansion parameter γ is small, one can expand the solutions of (62) around the free eigenmodes X_0, solutions of

$$\sin(rX_0)\cos\left((1-r)\frac{c_1}{c_2}X_0\right)+\frac{\rho_2 c_2}{\rho_1 c_1}\sin\left((1-r)\frac{c_1}{c_2}X_0\right)\cos(rX_0) = 0 \tag{63}$$

that are real numbers. Thus, one can write $X = X_0 + \delta X$, and obtain

$$\operatorname{Im}(\delta X) = \frac{\operatorname{Im}(Tr)\sin(rX_0)\cos(rX_0)}{r+\dfrac{\rho_1 c_1^2}{\rho_2 c_2^2}(1-r)+\alpha(1-r)\left(1-\dfrac{\rho_1 c_1}{\rho_2 c_2^2}\right)^2\cos^2(rX_0)}. \tag{64}$$

Thus, instability occurs if $\operatorname{Im}(Tr)\sin(rX_0)\cos(rX_0) < 0$, i.e. in the model considered $\sin(rX_0)\cos(rX_0) > 0$. For a given position of the heater in the tube, the growth rate is directly proportional to the imaginary part of the transfer function. This growth rate must be sufficiently large to overcome damping effects, which are acoustic radiation losses at the free ends of the tube, transverse heat conduction and friction at the tube walls. Thus, as can be seen from (60), the instability of a given mode will be favoured if γ is increased (increasing the heating power), or if the dimensionless velocity of the gas $U(c/LD)^{\frac{1}{2}}$ is neither too low nor too large.

4.2.3. *Transfer function for an expansion parameter γ of order unity*

Numerical treatment. We solve (47) with boundary conditions (48) numerically by using a shooting method (see for instance Press *et al.* 1986).

Far away at infinity in the fresh gases, the solution of (47) can be expanded as

$$\phi(\xi) \approx 1-\gamma\,\mathrm{e}^{\xi}+B\gamma\exp(\chi_+\xi), \tag{65}$$

and the complex number B is determined by the shooting method in order to satisfy the boundary condition $\phi'(0) = 0$.

The shooting method implements a Newton–Raphson method. One starts intregration of (36) from the point $\xi = -8$ where the solution (65) is used, and computes $\phi'(0)$ by using a fifth-order Runge–Kutta method with a constant step size ($h = 8/200$)). Other choices of the starting point taken in the interval $[-9, -4]$ do not change the result. The method is very efficient: only one or two shootings are required to adjust the complex number B and reach the condition $\phi'(0) = 0$ with an accuracy of 10^{-15}.

Results. In figures 1 and 2 the real and imaginary parts of the transfer functions are drawn for the small value of the expansion parameter $\gamma = 0.1$. The asymptotic transfer function given by (60) (curves 1) is compared with the transfer function obtained numerically (curves 2). A very good agreement between the two solutions is obtained: the discrepancy between the curves 1 and 2 is of order γ^2, which is consistent with the degree of approximation of (60).

In figures 3 and 4, the real and imaginary parts of the transfer function are drawn for the three values of the parameter γ 0.5, 1 and 5. For any value of γ, the transfer function takes the value γ (i.e. $\mathrm{Re}\,(Tr) = \gamma$ and $\mathrm{Im}\,(Tr) = 0$) for the zero frequency. This is simply obtained by writing the mass conservation $\rho_1 u_1 = \rho_2 u_2$. It is interesting to note that the extremum of the imaginary part of the transfer function is obtained for a value of $\omega\tau \approx 1$; the absolute value increases as γ increases, and the range of dangerous frequencies becomes smaller.

4.2.4. *Analytical solution in the case $\rho\nu$ constant*

A transfer function can be found exactly, when one assumes $\rho\nu$ = constant. In this case, one can solve analytically the system of equations (37)–(39) with the boundary conditions (6) and (7) by using the stream-function variable $\psi = \int_0^x \rho\,\mathrm{d}x$ (see for instance (9)). Equations (37)–(39) become

$$\frac{\partial\rho}{\partial t}+m_0\frac{\partial\rho}{\partial\psi}+\rho^2\frac{\partial u}{\partial\psi} = 0, \tag{66}$$

$$\frac{\partial T}{\partial t}+m_0\frac{\partial T}{\partial\psi}-\rho\nu\frac{\partial^2 T}{\partial\psi^2} = 0, \tag{67}$$

and

$$\rho T = \rho_1 T_1. \tag{68}$$

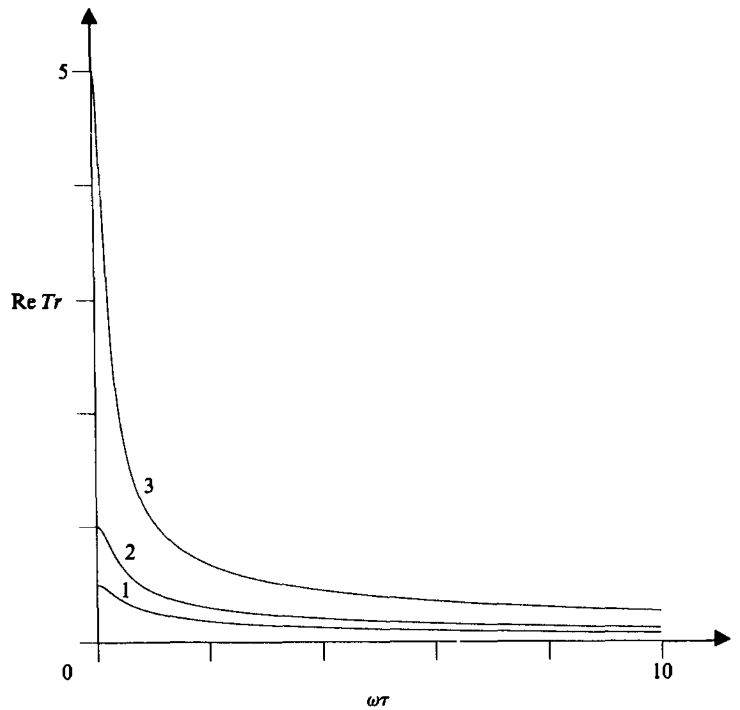

FIGURE 3. Real part of the transfer function – numerical result: curve 1, $\gamma = 0.5$; curve 2, $\gamma = 1$; curve 3, $\gamma = 5$.

Here c_p is assumed constant and $m_0 = (\rho u)_{x=0}$ is the mass flux at the grid. The steady solution is the same as the one given by relations (21)–(23) where the variable ξ defined in (20) is now $(m/\rho\nu)\psi$.

By using (66)–(68) the disturbances of temperature $\theta(\psi)\exp(\mathrm{i}\omega t)$, and velocity $\delta u(\psi)\exp(\mathrm{i}\omega t)$ of the steady solution satisfy

$$\mathrm{i}\omega\theta + m\frac{\mathrm{d}\theta}{\mathrm{d}\psi} - \rho\nu\frac{\mathrm{d}^2\theta}{\mathrm{d}\psi^2} = -\delta m_0\frac{\mathrm{d}\bar{T}}{\mathrm{d}\psi}, \tag{69}$$

$$\rho_1 T_1\frac{\mathrm{d}(\delta u)}{\mathrm{d}\psi} = \mathrm{i}\omega\theta + m\frac{\mathrm{d}\theta}{\mathrm{d}\psi} + \delta m_0\frac{\mathrm{d}\bar{T}}{\mathrm{d}\psi}. \tag{70}$$

Here δm_0 is the disturbance of the mass flux at the grid, and $m = \rho_1 U$ is the steady mass flow rate. By using the conditions $\theta(-\infty) = 0$, $\theta(+\infty) = 0$ and the steady solution, the solution of (69) is simply found as

$$\theta(\psi) = A\exp(\chi_+\psi) - \delta m_0\frac{T_2 - T_1}{\mathrm{i}\omega}\frac{m}{\rho\nu}\exp\left(\frac{m}{\rho\nu}\psi\right), \quad \psi < 0, \tag{71}$$

$$\theta(\psi) = B\exp(\chi_-\psi), \quad \psi > 0, \tag{72}$$

where

$$\chi_\pm = \frac{1}{\rho_1 l_{\mathrm{d}}}\frac{1 \pm (1 + 4\mathrm{i}\omega\tau)^{\frac{1}{2}}}{2}. \tag{73}$$

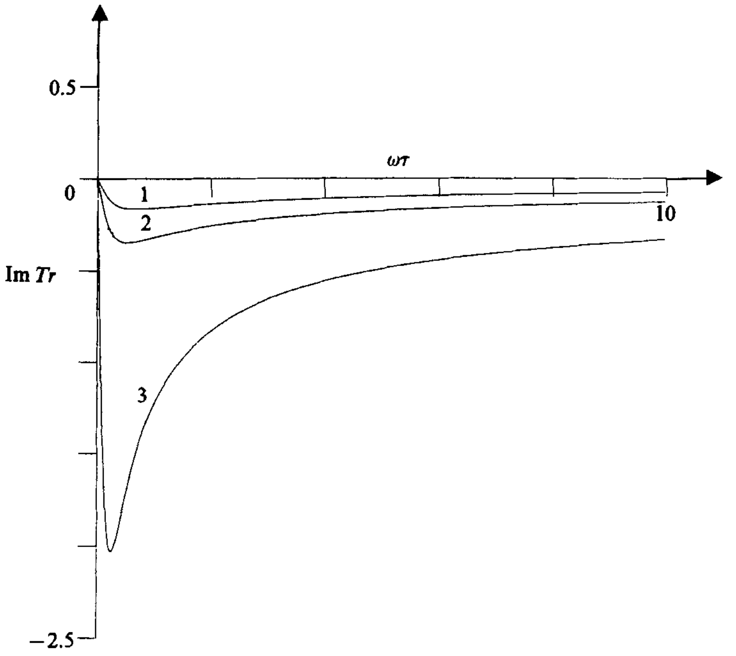

FIGURE 4. Imaginary part of the transfer function – numerical result: curve 1, $\gamma = 0.5$; curve 2, $\gamma = 1$; curve 3, $\gamma = 5$.

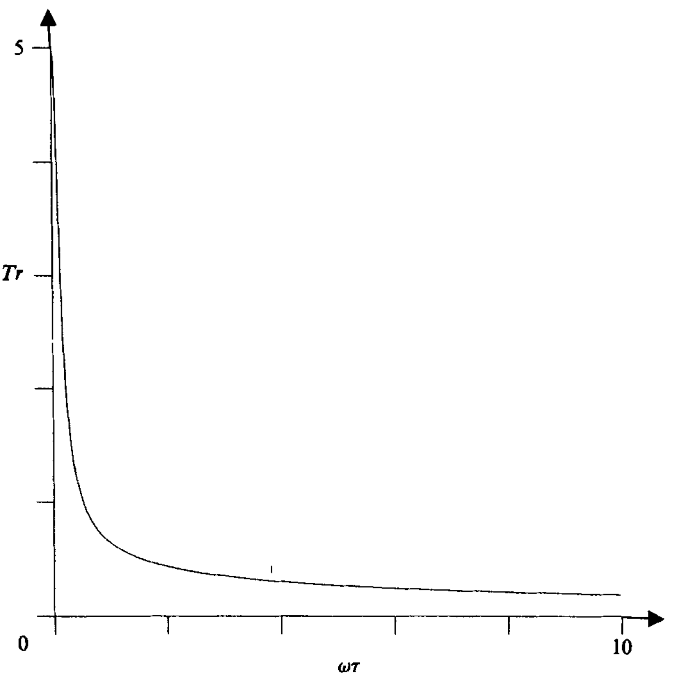

FIGURE 5. Real part of the analytical transfer function for $\gamma = 5$.

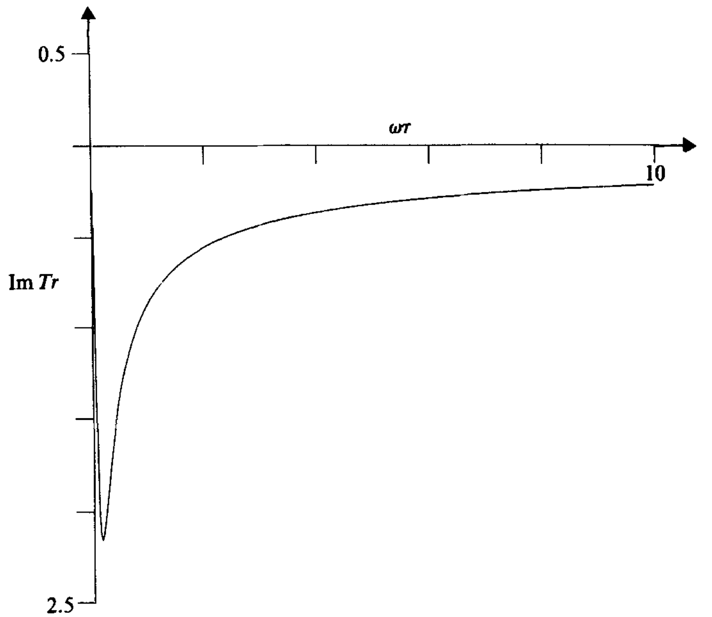

FIGURE 6. Imaginary part of the analytical transfer function for $\gamma = 5$.

Here $l_d = \rho\nu/(\rho_1^2 U)$ is the diffusive length and $\tau = l_d/U$ is the associated diffusive time. By using the boundary condition (6), one deduces $\theta(0) = 0$ and thus

$$\theta(\psi) = \delta m_0 \frac{T_2 - T_1}{i\omega} \frac{m}{\rho\nu} \left\{ \exp(\chi_+ \psi) - \exp\left(\frac{m}{\rho\nu}\psi\right) \right\}, \quad \psi < 0, \tag{74}$$

$$\theta(\psi) = 0, \qquad \psi > 0, \tag{75}$$

From (70) one obtains

$$\delta u(0) - \delta u_{-\infty} = \frac{\rho\nu}{\rho_1 T_1} \delta m_0 \frac{T_2 - T_1}{i\omega} \frac{m}{\rho\nu} \left(\chi_+ - \frac{m}{\rho\nu} \right), \tag{76}$$

from which follows the transfer function

$$Tr(\omega\tau) = \frac{2\gamma}{(1+\gamma)(1+(1+4i\omega\tau)^{\frac{1}{2}}) - 2\gamma}. \tag{77}$$

In the limit of small values of the parameter γ, the transfer function has the limiting expression given by (60), which confirms the fact that at small values of γ, the transfer function is independent of the function $\nu(\overline{T})$.

In figures 5 and 6 the real and imaginary parts of the transfer function are drawn for $\gamma = 5$. The comparison with the numerical integration allows us to note that the dependence of the function $\nu(\overline{T})$ on the temperature has little importance for the general form of the transfer function.

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